Direct Calculation of Ellipse Overlap Areas for Force-Based Models of Pedestrian Dynamics

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ABSTRACT

Computer simulations based on models of pedestrian dynamics have become useful tools for evaluating emergency egress scenarios. 'Microscopic' models track individual pedestrian movements, which are then used to describe macroscopic pedestrian flow. Simulations enable the comparison of pedestrian facilities designs, evaluation of escape routes in various scenarios, and the study of more theoretical questions. Space-continuous, force-based models simulate interactions between pedestrians based on their separation distance and relative velocities. In a common approach, pedestrian's 'sensory zones' are modelled as ellipses, with ellipse parameters varying dynamically according to the pedestrian's direction and velocity. Interactions between individual pedestrians, and between pedestrians and the environment, are controlled in part by overlapping of sensory zones; path deviations result from superposition of repulsive and driving forces. Implementation of an ellipse-based model requires efficient calculation of ellipse overlap areas. Historically, overlap areas have been estimated by approximating ellipse boundaries with polygons or other proxy curves. More recently, an approach for direct calculation of ellipse overlap area has been described, using an algorithm for determining the area of an ellipse segment. The segment algorithm is then used to calculate the overlap area between two general ellipses, using points of intersection between the two ellipses to identify appropriate segment areas. Intersection points can be found by solving the two implicit ellipse equations simultaneously. Recent innovations to the core algorithm include effective relative position determination, increasing both efficiency and robustness of the overlap area algorithm. This paper describes the direct overlap area algorithm. Implementations in C++ are compared for speed and accuracy with proxy curve approaches. Benefits of the direct algorithm are demonstrated within the context of a force-based model of pedestrian dynamics.

INTRODUCTION

Force-Based Models of Pedestrian Dynamics

Microscopic models of pedestrian dynamics specify the properties of individuals, and define their interactions. The definitions are then used to simulate movement of individual pedestrians within a crowd under specified environmental circumstances. The modeled crowd system is then analyzed using characteristics of the simulated pedestrian streams e.g., density vs. velocity. The realism of the model can be judged by comparing simulated crowd characteristics to observed crowd behavior, whenever the model and observations occur in similar environments.

Force-based models represent an important class of microscopic models for pedestrian dynamics. Observations indicate that pedestrians will deviate from a straight path when they are influenced by the presence of other pedestrians. A path deviation represents an acceleration of a pedestrian's motion, which according to Newton's laws implies the existence of an external force. Force-based models take Newton's second law of dynamics as a guiding principle and profit from a rich theory of dynamical systems and well-known numerical methods.

A Generalized Force-Based Model

The Generalized Centrifugal Force Model (GCFM) (Chraibi et al., 2010) describes the time evolution of individual pedestrians within a crowd by implementing a system of superposing short-range forces. Inter-pedestrian forces are governed by the spatial relationship between the "sensory-zone boundaries" of neighboring pedestrians as well as their relative velocities. Templer (1992) described the "sensory zone" as an elliptical space surrounding a pedestrian that is maintained, for psycho-cultural reasons, to avoid physical conflicts with other pedestrians and objects in the environment.

The GCFM uses ellipses to represent the two dimensional sensory-zone boundaries of individual pedestrians; ellipse axes are oriented in the instantaneous direction of travel, and semi-axes lengths are adjusted according to instantaneous velocity. Inter-pedestrian forces are derived from the distance between neighboring ellipses, d_{ij} in the left panel of Fig. 1. In some circumstances, such as high-density regions, sensory-zone ellipses may overlap, and in such cases the inter-pedestrian force is determined from the ellipse overlap area, illustrated in the right panel of Fig. 1. This paper describes an efficient method for calculating the overlap area of two general ellipses, which can be used to boost the computational efficiency of simulations using force-based pedestrian dynamics models.
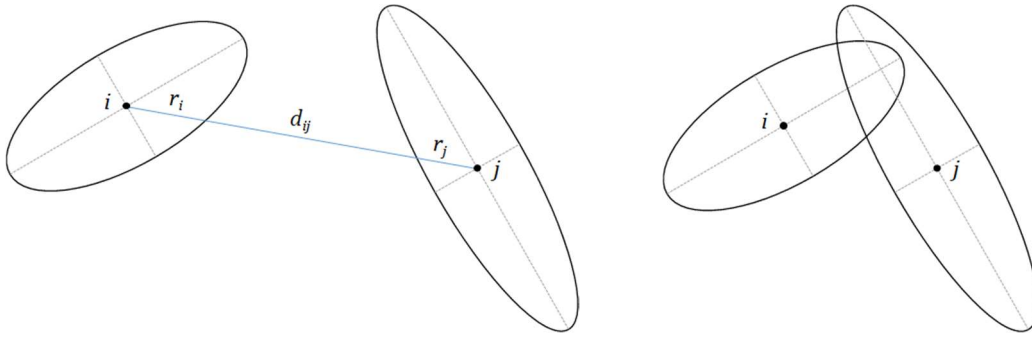

Figure 1*: In the GCFM, short-range forces between pedestrians are derived from the separation distance or the overlap area of sensory zones, which are represented by ellipses.*

NUMERIC APPROXIMATION OF ELLIPSE OVERLAP AREAS

Polygon Approximation

The area of an ellipse can be approximated by the finding the area of an inscribed polygon, calculated by summing the areas of adjacent triangles that comprise the polygon (Fig. 2, left panel).
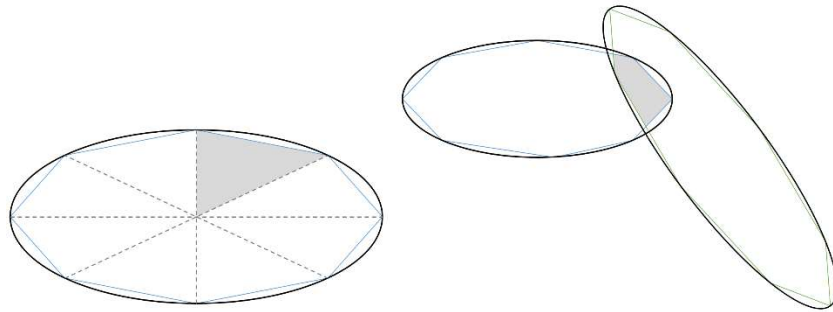

Figure 2*: <u>Left</u>: The area of an ellipse is approximated by an inscribed polygon. <u>Right</u>: The overlap area of two ellipses can be approximated by the shared area of two inscribed polygons.*

The polygon area asymptotically approaches the ellipse area as the number of vertices increases. The polygon area calculation requires evaluation of the ellipse equation at each vertex, so higher area precision becomes more computationally expensive. The polygon approach can be exploited to determine the overlap area of two intersecting ellipses. Intersection points of polygon edges are determined (e.g., O'Rourke et al., 1982; Toussaint, 1985), and provide demarcation points for finding the appropriate area representing the overlap.

Gauss-Green Area

An alternative method for determining ellipse overlap area is based on a direct calculation from the ellipse properties. Consider a general ellipse that may be rotated and translated, with semi-axis length A along the x-axis, and semi-axis length B along the y-axis. Then the ellipse is defined by a locus of points that satisfy the parametric expression in Eq. (1):

$$\left. \begin{array}{l} x(t) = A \cdot cos(\varphi) \cdot cos(t) - B \cdot sin(\varphi) \cdot sin(t) + h \\ y(t) = A \cdot sin(\varphi) \cdot cos(t) + B \cdot cos(\varphi) \cdot sin(t) + k \end{array} \right\} \quad 0 \le t \le 2\pi \qquad (1)$$

A rotated-then-translated ellipse can be defined by the set of parameters $\{A, B, h, k, \varphi\}$, with the understanding that the rotation through φ is performed before the translation through (h, k). The same general ellipse is written as an implicit polynomial by Eq. (2):

$$AA \cdot x^2 + BB \cdot x \cdot y + CC \cdot y^2 + DD \cdot x + EE \cdot y + FF = 0 \qquad (2)$$

The relationship between parametric constants $\{A, B, h, k, \varphi\}$ and the equivalent polynomial coefficients $\{AA, BB, CC, DD, EE, FF\}$ is given in Hughes and Chraibi (2014). The area of an ellipse sector between two points is swept out by a vector from the center to the first point (x_1, y_1) as the vector tip travels along the ellipse in a counter-clockwise direction to the second point (x_2, y_2), calculated using the Gauss-Green formula as in Eq. (3):

$$\text{Sector Area} = \frac{A \cdot B}{2} \int_{\theta_1}^{\theta_2} dt = \frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} \qquad (3)$$

The method of Eq. (3) for determining the area of an ellipse sector is illustrated in Fig. 3 (left panel). The overlap area between two ellipses is found by determining the sector areas in each ellipse between the intersection points, and adding or subtracting appropriate triangle areas, illustrated in Fig. 3 (right panel). The overlap area is the sum of the two shaded areas. Each shaded area can be determined from an ellipse sector, minus the triangular regions formed by the intersection points and the ellipse centers.
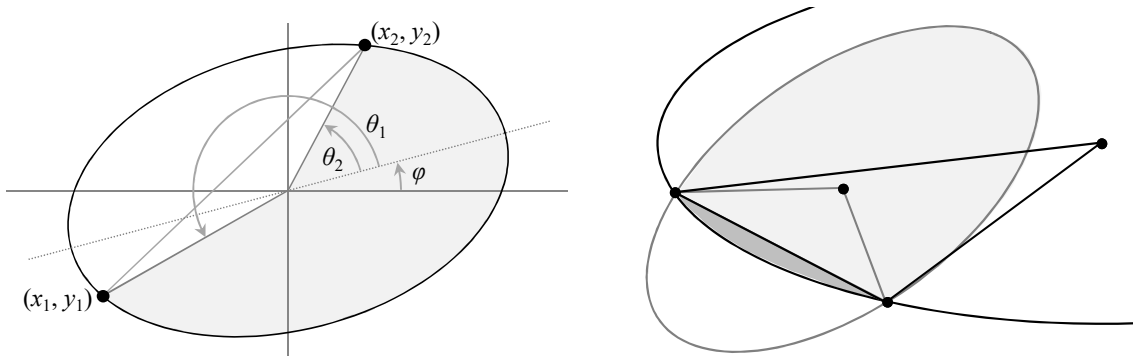

Figure 3: _Left_: The area of an ellipse sector between two points on the ellipse. _Right_: The overlap area between two ellipses from sector areas, and the addition or subtraction of appropriate triangular areas.

In order to implement Eq. (3), the intersection points between two overlapping ellipses must be found. In general, intersection points are found by solving the two implicit polynomials

simultaneously; in the most general case, the solution is found by determining the roots of a quartic polynomial.

Precision of the overlap area depends on the precision of the location of intersection points. Furthermore, the efficiency of the overlap area algorithm is affected by efficiency of the root-finding algorithm. Implementation of the analytical solution on an Intel Core i7-2620M, 2.7 GHz, 4 MB Cache, is 40 to 140 times faster than the solution based a polygon-approximation of ellipses with $n = 8$ and $n = 20$ vertices, respectively. For the cases tested, the overlap areas were known precisely; the relative error of areas determined by polygon approximation ranged from 0.1 to 0.01, respectively, while the direct method areas had a relative error of <1e-6. Additional details of the direct overlap area determination using the Gauss-Green method are presented in Hughes and Chraibi (2014).

Numeric Implementation for Pedestrian Dynamic Modeling

Simulations that implement force-based dynamics models based on elliptical sensory zones must address ellipse overlap areas for every pair of adjacent pedestrians in every frame. Efficiency of the area overlap algorithm thus becomes an important consideration for computation, particularly for extended simulations with many pedestrians. Regardless of the method used to determine ellipse overlap area, algorithm efficiency is improved by first determining the relative position of the two ellipses. Twelve distinct cases of the relative position between two ellipses are identified (Alberich-Carramiñana et al., 2017; Etayo et al., 2006), illustrated in Fig. 4.
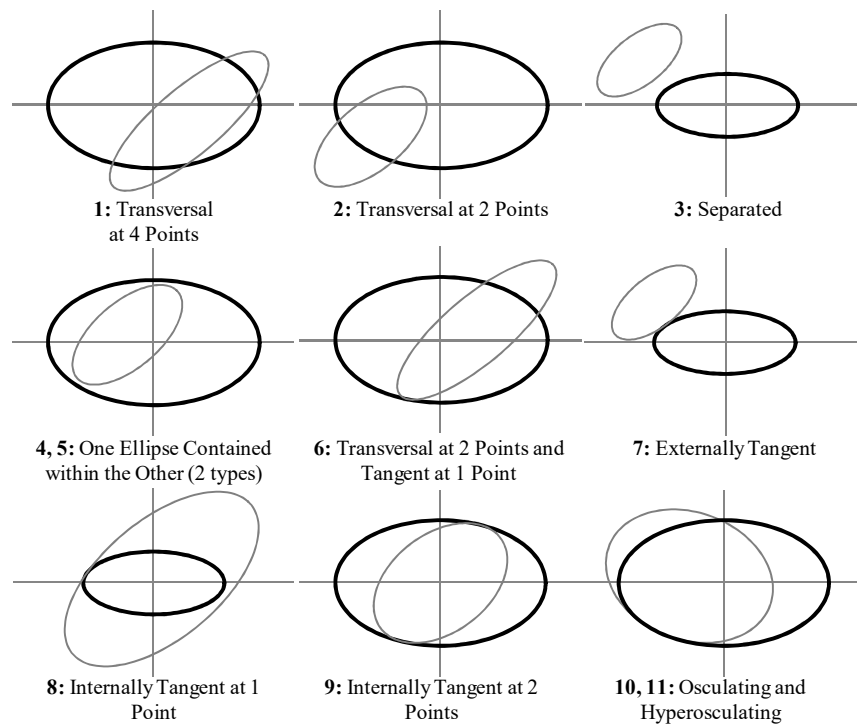

1: Transversal at 4 Points   **2:** Transversal at 2 Points   **3:** Separated

4, 5: One Ellipse Contained within the Other (2 types)   **6:** Transversal at 2 Points and Tangent at 1 Point   **7:** Externally Tangent

8: Internally Tangent at 1 Point   **9:** Internally Tangent at 2 Points   **10, 11:** Osculating and Hyperosculating

Figure 4*: Twelve distinct cases of the relative position of two ellipses. Case 0 (not shown) represents coincident ellipses. Only five of the 12 cases require calculation of non-trivial areas.*

Alberich-Carramiñana et al. (2017) describe an algorithm for determining the relative position of two general ellipses. A binary decision tree is formulated, whereby the decision at each node is based on the sign of algebraic expressions calculated from the two sets of implicit polynomial coefficients; the most complex expressions are third and second order polynomial discriminants. Determining overlap area of two ellipses is vastly more efficient when the relative position is determined prior to

implementing the search for intersection points. For example, if the two ellipses are externally tangent, the overlap area is zero, and there is no need to check for intersection points at all.

VALIDATION OF THE DIRECT OVERLAP AREA METHOD

Force-based pedestrian dynamics models that use polygon-based area algorithms may seek improved numerical efficiency by utilizing the direct ellipse area overlap method. The relative error of estimates produced by the two area algorithms differ; simulations using each of the two area methods with the same initial and boundary conditions will not produce the same pedestrian tracks. It is prudent to ask whether the choice of area method produces discernible differences in salient aspects of the modeled system.

Comparing system attributes for simulations using different area algorithms is akin to validating a model by comparing to observational data. One such validation tool is the Fundamental Diagram, a plot of density at various average velocities. A realistic model will display a density-velocity profile similar to experimental data collected in circumstances similar to the model environment.

Checking specifically for the effects of area algorithm choice on system dynamics, a similar metric is proposed based on a measure of spatial randomness of the point pattern. The Clark-Evans (CE) test statistic (e.g., Cressie, 1993) is a measure of complete spatial randomness. A list of discrete points in a continuous 2D domain, $\{(x_1, y_1), (x_2, y_2), …, (x_n, y_n)\}$ has a point density λ within the domain equal to $n/$area. Each point has a nearest neighbor at a specific distance, $\{d_1, d_2, …, d_n\}$. To calculate the CE test statistic, select a random sample of $m \approx 0.1\,n$ values from the set of nearest-neighbor distances. Then, the statistic of Eq. (4) follows a standard normal distribution:

$$Z_{CE} = \frac{\bar{d}_m - \frac{1}{2\sqrt{\lambda}}}{\sqrt{\frac{4-\pi}{4\pi m \lambda}}} \sim N(0,1) \tag{3}$$

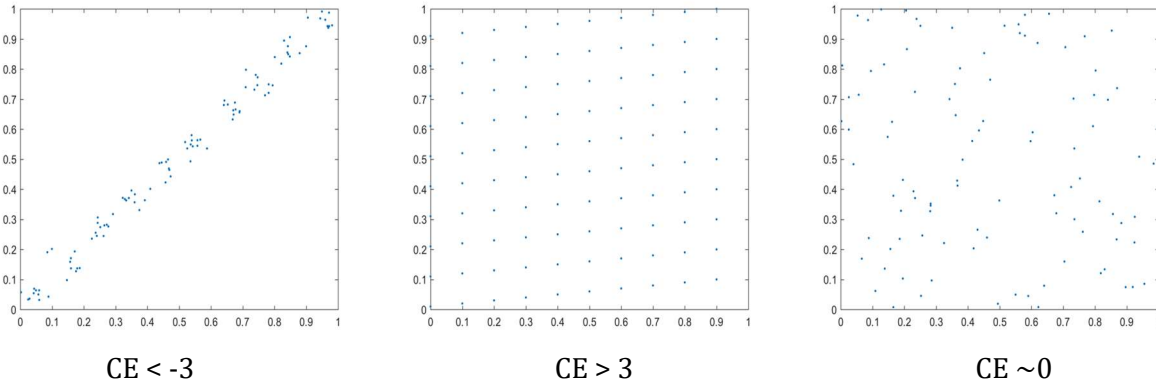$$\text{CE} < -3 \qquad\qquad \text{CE} > 3 \qquad\qquad \text{CE} \sim 0$$

Figure 5: *Spatial point patterns, and corresponding Clark-Evans test statistic ranges.*

Point patterns which display attributes of complete spatial randomness will have CE test statistics near zero, with a standard deviation of 1. One benefit if the CE test statistic is that the value can be used to characterize how a pattern might deviate from complete spatial randomness. Patterns that are significantly more clustered than random have highly negative CE values, whereas patterns that are significantly more ordered than random have highly positive CE values.

The CE test statistic can be used to compare force-based simulations with experimental data and for comparing simulations that use different area overlap algorithms. Fig. 6 illustrates use of CE test statistic for characterization of spatial randomness through a sequence of measured frames for a bi-directional corridor scenario from the BaSiGo experiment.
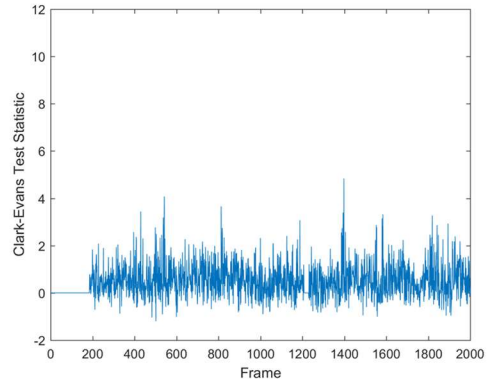
Figure 6*: CE test statistic through a sequence of measured frames for a bi-directional corridor scenario from the BaSiGo experiment. Project Description: http://www.basigo.de/ Data at: http://ped.fz-juelich.de/db/*

CONCLUSIONS

Force-based pedestrian dynamics models that utilize elliptical sensory zones must calculate the overlap area of adjacent ellipses. A commonly-used algorithm for determining overlap area uses inscribed polygons to approximate ellipse boundaries, and the ellipse overlap area is estimated by the shared area of the approximating polygons. A direct method for ellipse overlap area exploits the Gauss-Green formula, given any points of intersection between the two ellipses. The direct method is numerically more efficient with less relative error than the polygon approach. Either area method will benefit from a pre-characterization of relative position using an efficient decision tree algorithm. Validation of any changes in the choice of overlap area algorithm is aided by system-level characterization, such as comparison of density-velocity and spatial randomness-density profiles.

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